Finally, the solution of Eq. (11) will be

$$T(r,z) = T_a + Az - \frac{E_z}{4k'}(\alpha' A + \sigma E_z) r^2 \tag{24}$$

Boundary condition (13) and relation (16) give

$$W_{R_c} = (-\beta E_z - k'A) U_z + \frac{E_z}{2}(\alpha' A + \sigma E_z) R_c Ur \tag{25}$$

From this relation the expression

$$A = \left[-\left(2\beta E_z k' + \frac{E_z^4 \alpha' \sigma R_c^2}{2}\right) \Big/ 2\left(k' + \frac{E_z^2 \alpha'^2}{4}\right)\right]$$
$$\pm \left\{ \left[\left(2\beta E_z k' + \frac{E_z^4 \alpha' \sigma R_c^2}{2}\right)\right]^2 - 4\left[-W_{R_c} + \beta^2 E_z^2\right.\right.$$
$$\left.\left. + \frac{E_z^4 \sigma^2 R_c^2}{4}\right]\left(k'^2 + \frac{E_z^2 \alpha'^2}{4}\right)\right]^{1/2} \Big/ 2\left[k' + \frac{E_z \alpha'^2}{4}\right]\right\} \tag{26}$$

is obtained.

The temperature must decrease with increasing z—then the negative value of A is taken. Also the condition

$$\alpha' A + \sigma E_z > 0 \tag{27}$$

must exist, because the temperature decreases in the radial direction.

One may conclude that it is desirable in a cylindrical arc to make axial and radial temperature measurements in order to compare with the results from nonequilibrium thermodynamics modeling.

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Behavior of the Flow Through a Numerically Captured Shock Wave

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Introduction

AT present the most common method of predicting transonic flow is to solve numerically the potential equation representing conservation of mass using a shock capturing algorithm.[1,2] In a shock capturing algorithm the shock is represented by a finite velocity gradient spread over several mesh points, rather than by a discontinuity. Two types of finite difference algorithms are in use: nonconservative algorithms which do not conserve mass through the shock wave and conservative algorithms which do conserve mass. The issue of conservation or nonconservation of mass through the shock wave concerns the end points of the shock capture region. This Note concerns the question of mass conservation during the shock capture in a conservative algorithm.

The conclusion of this analysis is that a conservative algorithm merely conserves mass at the end points of the shock capture region. In the shock region there is an increase in mass followed by a decrease in mass.

This work was initiated to assist in a related investigation[3]; however, it is felt that the analysis also is of some interest in aiding understanding of the numerical algorithms used in transonic flow predictions.

Analysis

A simple description of transonic flow can be obtained by a study of the transonic small-disturbance equation, written in the form

$$[u - (u^2/2)]_x + v_y = 0 \tag{1}$$

where $u = (\gamma+1)M_\infty^2/(1-M_\infty^2)(\partial\phi/\partial x)$, $v = (\gamma+1)M_\infty^2/(1-M_\infty^2)^{3/2}(\partial\phi/\partial y)$, and ϕ is the perturbation potential. In this formulation $u>1$ denotes a supersonic flow and $u<1$ denotes a subsonic flow. Equation (1) can give discontinuous solutions, which for a normal shock gives the jump relation

$$u_1 + u_2 = 2 \tag{2}$$

It is usually the case that Eq. (1) is solved by a finite difference scheme with an artificial dissipation added in the supersonic domain by using an upwind difference procedure,[1,2] and in the following analysis it is assumed that upwind differencing is used throughout. One-dimensional examples have been computed[2] using this method.

The upwinding scheme of Murman[1] can be represented by a differential equation of the form that approximates Eq. (1) by

$$\frac{\partial}{\partial x}\left[\left(u - \frac{u^2}{2}\right) - \Delta x\left(u - \frac{u^2}{2}\right)_x\right] + v_y = 0 \tag{3}$$

where Δx is the grid size. Equation (3) can be integrated to give

$$u - u^2/2 = P(x,y) + C_2(y)e^{x/\Delta x} \tag{4}$$

where $V(x,y)$ is given by

$$V(x,y) = \int^x v_y \mathrm{d}x \tag{5}$$

where

$$P(x,y) = e^{x/\Delta x}\int^x [V(x,y) + C_1(y)]e^{-(x/\Delta x)}/\Delta x \mathrm{d}x \tag{6}$$

and $C_1(y)$ and $C_2(y)$ are arbitrary functions arising from the integration. $C_2(y)$ is also a function of Δx such that

$$\lim_{\Delta x \to 0} C_2(y)e^{x/\Delta x} \to 0$$

in order to recover the correct solution of Eq. (3) when $\Delta x = 0$. If shock waves are normal to the x axis then V and C_1 are continuous functions of (x,y). Since $V(x,y)$, $C_1(y)$ are continuous functions, $P(x,y)$ is a continuous function; and since shocks are assumed normal to the x axis, $C_2(y)$ is a continuous function.

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Consider now the quasi-one-dimensional problem where V is a function only of y. Equation (4) then becomes

$$u - u^2/2 = P(y) + C_2(y)e^{x/\Delta x} \quad (7)$$

Consider now the shock capture region, starting at $x = x_0$ and a series of discrete points at $x_n = x_0 + n\Delta x$, where $n = 0,1,2,\ldots$. It then follows that

$$u_n - u_n^2/2 = P + C_2 e^{x_0/\Delta x} e^n \quad (8)$$

Hence

$$u_n - u_n^2/2 = u_0 - (u_n^2/2) + C_2 e^{x_0/\Delta x}(e^n - 1) \quad (9)$$

Equation (9) can be solved to give

$$u_n = 1 \pm \{1 - 2[u_0 - (u_0^2/2)] - 2C_2 e^{x_0/\Delta x}(e^n - 1)\}^{1/2} \quad (10)$$

The positive root is always supersonic and the negative root is subsonic. In the shock capture region the solution must jump between a positive root and a negative root in order to decelerate from supersonic to subsonic speeds.

If it is assumed that

$$C_2 e^{x_0/\Delta x} \ll (1 - u_0)^2 \quad (11)$$

then Eq. (10) simplifies to

$$u_n = 1 \pm (u_0 - 1) \mp \frac{C_2 e^{x_0/\Delta x}(e^n - 1)}{(u_0 - 1)} \quad (12)$$

If the second point on the shock capture region, u_1, is supersonic, then the positive root is taken and

$$u_1 = u_0 - \frac{C_2 e^{x_0/\Delta x}(e - 1)}{(u_0 - 1)} \quad (13)$$

If $u_1 < u_0$, Eq. (13) indicates that $C_2 > 0$.

If the next point is subsonic, the negative root must be taken and

$$u_2 = 2 - u_0 + \frac{C_2 e^{x_0/\Delta x}(e^2 - 1)}{(u_0 - 1)} \quad (14)$$

Once this jump has taken place the solution must stay on the subsonic root. This shock jump relation is obtained from Eqs. (17) and (18) to be

$$u_1 - u_2 = 2u_0 - 2 - \frac{C_2 e^{x_0/\Delta x}(e + e^2 - 2)}{(u_0 - 1)} \quad (15)$$

compared to the true theoretical jump of

$$u_1 - u_2 = 2(u_0 - 1) \quad (16)$$

Thus, since $C_2 > 0$ the shock capture does have a jump but of a magnitude less than the weak jump of the equation. The rest of the true jump condition is composed of a supersonic deceleration and a subsonic deceleration. One sequence in a shock capture region is as follows: 1) smooth supersonic deceleration; 2) jump from supersonic to subsonic between grid points; 3) smooth subsonic deceleration to complete the shock compression.

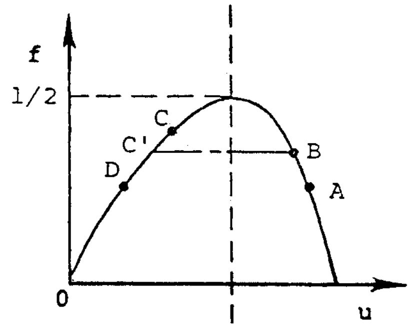

Fig. 1 Variation of f through a shock.

Table 1 Variations of ρU through a shock wave. One-dimensional algorithm[2] (no switching).

ϕ_x	ρU	ϕ_x	ρU
1.40	0.5208	1.052	0.6319
1.395	0.5235	0.8893	0.6246
1.385	0.5287	0.7384	0.5819
1.365	0.5389	0.65	0.5416
1.322	0.559	0.625	0.5235
1.216	0.5994	0.6125	0.5212

If the quantity $[u - (u^2/2)]$ is conserved through a shock wave, then it follows that for a given u_0 there is only one other solution, $\bar{u}_0$, which conserves $[u - (u^2/2)]$ which is

$$\bar{u}_0 = 2 - u_0 \quad (17)$$

However, in a shock capture solution there are several points between these theoretical end points and hence in the shock capture $[u - (u^2/2)]$ cannot be conserved, although mass is conserved at the shock end point.

Consider the function f, where

$$f = u - u^2/2$$

The variation of f with u is sketched in Fig. 1. The points ABCD in Fig. 1 are the shock capture points in a finite difference solution, i.e., a series of values of u decreasing from u_0 at A to u_3 at D. Now, in order to get from A to D, f must increase to reach the value at B by smooth supersonic deceleration. To get from B to C the solution jumps from the supersonic root to the subsonic root as described earlier. This jump takes place somewhere between B and C, say a point C′. If the value at C is greater than the value at C′ then f is increased further to the value at C. Finally, f is decreased back to its original value at D by smooth subsonic deceleration.

In a full potential solution a similar process occurs with the flux f identified as the streamwise mass flux (ρU). Numerical results computed from a one-dimensional solution[2] of the algorithm used in the code developed by Holst[4] are shown in Table 1, and it can be seen that the above hypothesis is confirmed.

Hence, it can be seen that the shock capture region in a finite difference algorithm is artificial and no great significance should be attached to its features.

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